Light–cone Superstring Field Theory, pp–wave background and integrability properties

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Abstract. We show that the three strings vertex coefficients in light–cone open string field theory satisfy the Hirota equations for the dispersionless Toda lattice hierarchy. We show that Hirota equations allow us to calculate the correlators of an associated quantum system where the Neumann coefficients represent the two–point functions. We consider next the three strings vertex coefficients of the light–cone string field theory on a maximally supersymmetric pp–wave background. Using the previous results we are able to show that these Neumann coefficients satisfy the Hirota equations for the full Toda lattice hierarchy at least up to second order in the 'string mass' μ.

1   Introduction

In recent years it has become more and more evident that integrability plays an important role in string theory. This fact is conspicuous in the $AdS_5 \times S^5$ example of AdS/CFT duality, where integrability features on both sides of the correspondence. Integrability

plays a major role in connection with topological strings. But there are also other less well–known cases. One of these is represented by the integrability properties of the Neumann coefficients for the three strings vertex of string field theory (SFT). In [1] it was shown that these Neumann coefficients in the case of Witten's covariant open bosonic string field theory (OSFT), [2], satisfy the Hirota equations, [3], of the *dispersionless* Toda lattice hierarchy [4, 5, 6]. This is to be traced back to the existence of a conformal mapping [7, 8] underlying the three strings vertex. The conclusions of [1] were limited to covariant OSFT. What we would like to show in this paper is that the above integrability properties seem to be a general characteristic of the three strings vertex. Indeed, below, we will prove that it holds for the light–cone string field theory (LCSFT) in flat background. But, what is more important, we will present evidence that it may also hold for a nontrivial background. We will specifically examine the Neumann coefficients of the three strings vertex in a maximally supersymmetric pp–wave background and expand them in terms of the 'string mass' μ. We will show that, up to second order in this expansion, they satisfy the (correspondingly expanded) Hirota equations for the *dispersive* Toda lattice hierarchy (i.e. for the full Toda lattice hierarchy), [9]. This leads us to the conjecture that the Neumann coefficients in this background satisfy the Hirota equations of the full Toda lattice hierarchy.

The results of this paper are admittedly of mere theoretical interest. They do not have an immediate practical impact. And perhaps they are not unexpected. Once the generating function of the Neumann coefficients are written in terms of conformal mappings (see below), they are expected to satisfy some kind of integrability condition, [8]. That this is true also for the three strings vertex in the pp–wave background may be seen as the consequence of the solvability of string theory on such background. However the main value of our results is that they point towards the existence of an integrable model (of which the Hirota equations are a signal) underlying the SFT structure, very likely a matrix model, which, once uncovered, would greatly improve our knowledge of SFT. The fact that this is probably true also for a nontrivial background, such as the pp–wave one, may suggest a way to approach the problem of defining SFT on more general backgrounds.

This paper is organized as follows. In section 2 we introduce the Neumann coefficients for LCSFT, define their generating functions and the conformal maps they can be derived from, and study their properties. In section 3 we show that these generating functions satisfy the Hirota equations for the dispersionless Toda lattice hierarchy. In section 4 we define the Hirota equations for correlators with more than two insertions. In section 5 we introduce the Neumann coefficients in a pp-wave background and prove that they satisfy the Hirota equations for the full Toda lattice hierarchy up to second order in μ.

2   The conformal maps for the light–cone SFT

In this section we work out the conformal properties of the three strings vertex in light–cone SFT. We introduce first the relevant Neumann coefficients and then we show that, like in Witten's OSFT, they can be defined in terms of conformal mappings and determine

them explicitly. We show how our formulas are related to the ones of Mandelstam, [10]. Finally we discuss the general features of these conformal mappings and describe an explicit example.

2.1 Neumann coefficients for bosonic three strings vertex

In LCSFT the three superstrings vertex is determined entirely in terms of the Neumann coefficients of the bosonic part. Therefore we will concentrate on the latter, which can be written in the form

$$|V_3\rangle = \int \prod_{s=1}^{3} d\alpha_s \prod_{I=1}^{8} dp_{(r)}^I \delta(\sum_{r=1}^{3} \alpha_r)\delta(\sum_{r=1}^{3} p_{(r)}^I)\exp(-\Delta_B)\,|0,p\rangle_{123} \tag{1}$$

where $I = 1,\ldots,8$ label the transverse directions, $\alpha_r = 2p_{(r)}^+$ and

$$\Delta_B = \sum_{r,s=1}^{3} \delta_{IJ} \left(\sum_{m,n\geq1} a_m^{(r)I\dagger}V_{mn}^{rs}a_n^{(s)J\dagger} + \sum_{n\geq1} p_{(r)}^I V_{0n}^{rs}a_n^{(s)J\dagger} + p_{(r)}^I V_{00}^{rs}p_{(s)}^J \right) \tag{2}$$

where summation over I and J is understood. The operators $a_n^{(s)J}, a_n^{(s)J\dagger}$ are the non–zero mode transverse oscillators of the s–th string. They satisfy

$$[a_m^{(r)I}, a_n^{(s)J\dagger}] = \delta_{mn}\delta^{IJ}\delta^{rs}$$

$p_{(r)}$ is the transverse momentum of the r–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacua relative to the three strings. $|p_{(r)}\rangle$ is annihilated by $a_n^{(r)}$ and is the eigenstate of the operator $\hat{p}_{(r)}^I$ with eigenvalue $p_{(r)}^I$. The vertex coefficients V_{nm} are more conveniently expressed in terms of the Neumann coefficients N_{nm} as follows:

$$V_{mn}^{rs} = -\sqrt{mn}N_{mn}^{rs}, \qquad V_{m0}^{rs} = \frac{\sqrt{m}}{6}\epsilon^{stu}\frac{\alpha_t - \alpha_u}{\alpha_r}N_m^r \equiv \frac{1}{\sqrt{m}}N_{m0}^{rs} \tag{3}$$

$$V_{00}^{rr} = \frac{1}{3}(\alpha_{r-1}^2 + \alpha_{r+1}^2)\frac{\tau_0}{\alpha}, \qquad V_{00}^{rs} = -\frac{2}{3}\alpha_r\alpha_s\frac{\tau_0}{\alpha}, \qquad r \neq s\ . \tag{4}$$

In these equations and in the sequel $r,s = 1,2,3$ modulo 3. In eq.(3) ϵ denotes the completely antisymmetric tensor with $\epsilon^{123} = 1$ and the indices t,u are summed over. In these equations the N's are related to the Mandelstam $\bar{N}$ by $N_{mn}^{rs} = \bar{N}_{mn}^{rs}$ and $N_m^r = \alpha_r\bar{N}_m^r$, with, see [10, 11],

$$N_m^r = \frac{\Gamma(-\frac{\alpha_{r+1}}{\alpha_r}\,m)}{m!\,\Gamma(-\frac{\alpha_{r+1}}{\alpha_r}\,m - m + 1)}\,e^{\frac{\tau_0}{\alpha_r}m}, \tag{5}$$

$$N_{mn}^{rs} = -\frac{\alpha\,m\,n}{\alpha_r\alpha_s(m\,\alpha_s + n\,\alpha_r)}N_m^r N_n^s \tag{6}$$

and

$$\alpha = \alpha_1\alpha_2\alpha_3, \qquad \tau_0 = \sum_{r=1}^{3}\alpha_r\,\ln|\alpha_r|, \qquad \sum_{r=1}^{3}\alpha_r = 0\ . \tag{7}$$

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The three strings vertex V_{m0}^{rs} and V_{00}^{rs} are not exactly the same as in [11]. We have symmetrized them using momentum conservation.

The Neumann coefficients (5–6) are homogeneous functions of the parameters α_r, so they actually depend on a single parameter β

$$\beta_r := \frac{\alpha_{r+1}}{\alpha_r}, \quad \beta_1 \beta_2 \beta_3 = 1, \quad \beta_r(\beta_{r+1} + 1) = -1, \tag{8}$$

$$\beta_1 = \beta, \quad \beta_2 = -\frac{\beta+1}{\beta}, \quad \beta_3 = -\frac{1}{\beta+1}, \tag{9}$$

$$-\infty < \beta_r \le -1, \quad -1 \le \beta_{r+1} \le 0, \quad 0 \le \beta_{r+2} < \infty,$$

$$e^{\frac{\tau_0}{\alpha_1}} = \frac{|\beta|^\beta}{|\beta+1|^{\beta+1}}, \quad e^{\frac{\tau_0}{\alpha_2}} = \frac{|\beta|}{|\beta+1|^{\frac{\beta+1}{\beta}}}, \quad e^{\frac{\tau_0}{\alpha_3}} = \frac{|\beta+1|}{|\beta|^{\frac{\beta}{\beta+1}}} \ . \tag{10}$$

We can organize the Neumann coefficients by means of generating functions

$$N^r(z) := \sum_{n=1}^\infty \frac{1}{z^n} N_n^r, \qquad N^{rs}(z_1, z_2) := \sum_{n,m=1}^\infty \frac{1}{z_1^n} \frac{1}{z_2^m} N_{nm}^{rs} \ . \tag{11}$$

Our first purpose is to express these functions in terms of conformal mapping from the unit semidisk to the complex plane, in analogy the case of covariant OSFT, [1].

We write the conformal mappings for LCSFT as follows

$$f_r(z^{-1}) = f_{r+2}(0) - \frac{(f_{r+2}(0) - f_r(0))(f_{r+1}(0) - f_{r+2}(0))}{(f_{r+1}(0) - f_r(0))\varphi_r(z^{-1}) + f_r(0) - f_{r+2}(0)} \tag{12}$$

with $f_1(0) \ne f_2(0) \ne f_3(0)$. The functions $\varphi_r(z^{-1}) \equiv \varphi_r$ are solutions to the equations

$$\varphi_r^{\beta_r}(\varphi_r - 1) = \frac{1}{z} e^{\frac{\tau_0}{\alpha_r}} := x_r \ . \tag{13}$$

We remark that, as a consequence of (12–13), we have the following identities

$$f_r'(0) = e^{\frac{\tau_0}{\alpha_r}} \frac{(f_{r+1}(0) - f_r(0))(f_{r+2}(0) - f_r(0))}{f_{r+1}(0) - f_{r+2}(0)},$$

$$\frac{f_r'(0) f_s'(0)}{(f_r(0) - f_s(0))^2} = e^{\tau_0(\frac{1}{\alpha_r} + \frac{1}{\alpha_s})}, \quad r \ne s \ . \tag{14}$$

By means of the conformal mappings we now define the generating functions

$$N^r(z) = -\beta_{r+2} \ln\left(-\frac{f_r'(0)}{z}\left(\frac{1}{f_r(0) - f_r(z^{-1})} + \frac{1}{f_{r+1}(0) - f_r(0)}\right)\right), \tag{15}$$

$$N^{rr}(z_1, z_2) = \ln\left(\frac{f_r'(0)}{z_1 - z_2}\left(\frac{1}{f_r(0) - f_r(z_2^{-1})} - \frac{1}{f_r(0) - f_r(z_1^{-1})}\right)\right), \tag{16}$$

$$N^{rs}(z_1, z_2) = \ln\left(\frac{(f_r(z_1^{-1}) - f_s(z_2^{-1}))(f_r(0) - f_s(0))}{(f_r(0) - f_s(z_2^{-1}))(f_r(z_1^{-1}) - f_s(0))}\right), \quad r \ne s \ . \tag{17}$$

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We notice that these definitions are very close to those in Witten's SFT, [1].

Let us analyze eq. (13). First of all we notice that any solution will have a branch that can be expanded around the value 1 as $1 + x_r + \ldots$

$$
\begin{aligned}
\varphi_r \;\equiv\; \varphi_r(z^{-1}) \equiv \varphi_r(x_r) &= 1 + x_r + \sum_{k=2}^{\infty} a_k x_r^k \\
&= 1 + x_r - \beta_r x_r^2 + \frac{1}{2}\beta_r(1 + 3\beta_r)x_r^3 - \frac{1}{3}\beta_r(1 + 2\beta_r)(1 + 4\beta_r)x_r^4 \\
&\quad + \frac{1}{24}\beta_r(1 + 5\beta_r)(2 + 5\beta_r)(3 + 5\beta_r)x_r^5 + \ldots \quad .
\end{aligned}
\tag{18}
$$

Using these expansions inside the definitions (15–17) one can make a direct comparison and verify that they do generate the Neumann coefficients of Mandelstam (5–6).

It must be clarified that the three functions $\varphi_1, \varphi_2, \varphi_3$ are not distinct. With very simple manipulations one can see that the solutions to eqs.(13) with different r are related by the following transformations:

$$
\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r\left(-x_{r+1}^{\beta_r}\right)} = 1 - \frac{1}{\varphi_{r-1}\left((-x_{r+1})^{\frac{1}{\beta_{r+1}}}\right)}.
\tag{19}
$$

Therefore the three equations (13) give rise to a unique solution (which describes a Riemann surface, see below).

The importance of these transformations should not be underestimated. The three strings vertex represents the fusion of two strings that come together and give rise to a third one. In Witten's covariant OSFT this process is very symmetric: two strings evolving from $\tau = -\infty$ come together at $\tau = 0$ in such a way that the right half of one string overlap with the left half of the other; the three strings vertex describes this process which ends with the emergence of the third string. The three strings vertex in the present case (LCSFT), as we shall see, has a different stringy/geometric interpretation. It is however still characterized by precise overlapping conditions that are made possible by the above equations.

Let us analyze the gluing conditions for the mappings $f_r(z_r^{-1})$ (12), i.e. let us see how we can satisfy the conditions

$$
f_{r+1}(z_{r+1}^{-1}) = f_r(z_r^{-1}) .
\tag{20}
$$

They lead to

$$
\varphi_{r+1}(x_{r+1}) = \frac{1}{1 - \varphi_r(x_r)}
\tag{21}
$$

which can easily be solved if one compares eqs. (21) and (19)

$$
x_r = -x_{r+1}^{\beta_r}.
\tag{22}
$$

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Equivalently, we have the following relations for the string coordinates:

$$z_r = -z_{r+1}^{\beta_r} \ . \tag{23}$$

For later use we record that on the unit circle these conditions become

$$z_r = e^{i\theta_r}, \quad \theta_r = \beta_r \, \theta_{r+1} + \pi. \tag{24}$$

Starting from the definitions (15–17) and using (8), (12) and (13), one can derive the following explicit representations

$$
\begin{aligned}
N^r(z) &= \frac{1}{\beta_r} \ln\left(-\frac{f_r'(0)}{z}\left(\frac{1}{f_r(0)-f_r(z^{-1})}+\frac{1}{f_{r+2}(0)-f_r(0)}\right)\right) \\
&\equiv \ln \varphi_r(z^{-1}) \equiv -\beta_{r+2} \ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\left(\frac{1}{1-\varphi_r(z^{-1})}-1\right)\right) \\
&\equiv \frac{1}{\beta_r}\ln\left(-\frac{e^{\frac{\tau_0}{\alpha_r}}}{z}\frac{1}{1-\varphi_r(z^{-1})}\right),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
N^{rr}(z_1,z_2) &= \ln\left(\frac{1}{z_1-z_2}\left(z_1\varphi_r^{\beta_r}(z_1^{-1})-z_2\varphi_r^{\beta_r}(z_2^{-1})\right)\right) \\
&\equiv \ln\left(\frac{e^{\frac{\tau_0}{\alpha_r}}}{z_1-z_2}\left(\frac{1}{1-\varphi_r(z_2^{-1})}-\frac{1}{1-\varphi_r(z_1^{-1})}\right)\right),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
N^{rr+1}(z_1,z_2) &= \ln\left(\frac{1}{\varphi_r(z_1^{-1})}+\varphi_{r+1}(z_2^{-1})-\frac{\varphi_{r+1}(z_2^{-1})}{\varphi_r(z_1^{-1})}\right) \\
&\equiv \ln\left(1+\frac{(1-\varphi_r(z_1^{-1}))(1-\varphi_{r+1}(z_2^{-1}))}{\varphi_r(z_1^{-1})}\right), \\
N^{rs}(z_1,z_2) &= N^{sr}(z_2,z_1)
\end{aligned}
\tag{27}
$$

which actually do not depend on the values of the conformal mappings (12) at the origin, $f_r(0)$, as long as they are distinct, i.e. $f_1(0) \neq f_2(0) \neq f_3(0)$. Therefore the parameters $f_r(0)$ in (12) are inessential gauge parameters.

2.2 A comparison with the Neumann function method

Eqs. (13) are related to the ones that appear in the derivation of open string tree amplitudes by means of the Neumann function method. At the tree level the interaction of open strings consists of the joining of two strings by the endpoints to form a unique string or the splitting of a string into two. These two processes can geometrically be described, [10, 12, 13], in terms of cutting and pasting string world–sheet strips. In turn such a geometry can be nicely represented in the complex plane by means of logarithmic

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conformal mappings. This leads to the explicit evaluation of the Neumann coefficients for the interaction of three strings that are precisely those introduced at the beginning of section 2. For instance, a suitable logarithmic map for the r–th three–string configuration is

$$\rho_r = \alpha_{r+1}\ln(z'-1) + \alpha_{r-1}\ln z' \tag{28}$$

where z' takes values in the upper half plane. This can be transformed into, see [12, 13],

$$y_r = -\beta_r\ln(1 + x_r e^{y_r}) . \tag{29}$$

This equation can easily be reduced to the form (13) if we set $\varphi_r = \exp(-\beta_{r+1}\beta_{r+2}y_r)$ and make the identification

$$x_r \equiv \frac{e^{\frac{\tau_0}{\alpha_r}}}{z} = -z'^{\frac{1}{\beta_{r-1}}}(z'-1)^{\beta_r} . \tag{30}$$

This means that our variable z in (13) is a 'uniformizing' variable for the three equations. To clarify this issue, in the next subsection, we work out an explicit example.

2.3 Particular case $\beta = 1$

Let us study in detail a particular case, specified by values of α_r for which the parameter $\beta = 1$. This case turns out to be particularly simple and can be analyzed in full detail. In this case $\beta_1 = 1, \beta_2 = -2, \beta_3 = -\frac{1}{2}$ and eqs. (13) can easily be solved. For each φ_r we have two branches

$$\varphi_1^{\pm}(z^{-1}) = \frac{1 \pm \sqrt{1 + \frac{1}{z}}}{2}, \tag{31}$$

$$\varphi_2^{\pm}(z^{-1}) = \frac{2}{1 \pm \sqrt{1 - \frac{1}{z}}}, \tag{32}$$

$$\varphi_3^{\pm}(z^{-1}) = \left(\frac{1}{z} \pm \sqrt{1 + \frac{1}{z^2}}\right)^2 . \tag{33}$$

The $^{(+)}$ branch is the one for which the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \ldots$. We can now define the corresponding conformal mappings. To specify the latter we need to fix the values they take at the origin. We recall that these values can be arbitrary as long as they are distinct. Therefore we choose them simply as

$$f_1(0) = -1, \qquad f_2(0) = 0, \qquad f_3(0) = 1 .$$

With this choice we get

$$f_1^{(\pm)}(1/z) = -\frac{1 \pm \sqrt{1 + 1/z}}{3 \mp \sqrt{1 + 1/z}} , \tag{34}$$

$$f_2^{(\pm)}(1/z) = \frac{\sqrt{1 - 1/z} \mp 1}{\sqrt{1 - 1/z} \pm 3} \ , \tag{35}$$

$$f_3^{(\pm)}(1/z) = -\frac{1}{1 - 2(1/z \pm \sqrt{1 + (1/z)^2})^2} \ . \tag{36}$$

Each couple of functions $f_r^{(\pm)}(1/z)$ defines a Riemann surface represented by two sheets joined through a cut. In the first case the cut runs from -1 to ∞, in the second it runs from 1 to ∞ and in the third between i and $-i$. These three Riemann surfaces however are related by the maps (19).

In the $\beta = 1$ case it is rather easy to see in detail the gluing conditions that characterize the three strings vertex. We consider three unit semidisks S_1, S_2 and S_3 cut out in the complex $\zeta_1 = 1/z_1$, $\zeta_2 = 1/z_2$ and $\zeta_3 = 1/z_3$ upper half planes. Each function $f_r^{(\pm)}$ maps the semidisk into a region $D_r^{(\pm)}$ in the image complex plane. $D_1^{(\pm)}$ and $D_2^{(\pm)}$ have the form of lobes, while $D_3^{(\pm)}$ are the outer part of a compact bi–lobed domain in the lower $(+)$ and upper $(-)$ half plane, respectively (see Fig. 1–3, where the $D_r^{(+)}$'s are shown, the $D_r^{(-)}$'s being the same regions reflected with respect to the real axis).
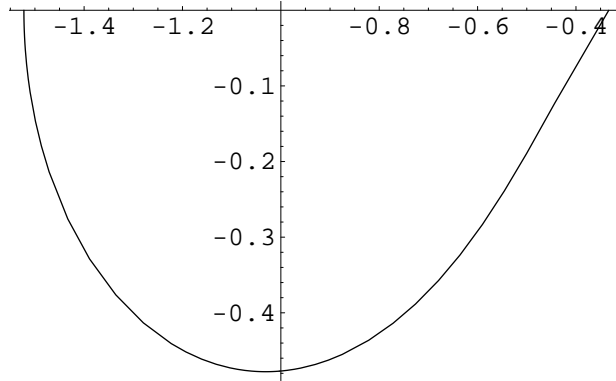

Figure 1: *The region $D_1^{(+)}$ is contained between the curve and the real axis.*

These domains are glued together along the borderlines in the following way. To start with, $f_1^{(+)}(\zeta_1) = f_2^{(-)}(\zeta_2)$ if $\zeta_1 = -\zeta_2$. This means that S_1 and S_2 are glued together along the real axis with opposite orientation. The image common boundary in $D_1^{(+)}$ and $D_2^{(-)}$ stretches also along the real axis. The second overlap condition we consider is between $D_1^{(+)}$ and $D_3^{(+)}$: $f_1^{(+)}(\zeta_1) = f_3^{(+)}(\zeta_3)$ if $\zeta_3 = -1/\sqrt{\zeta_1}$. This means the absolute values of ζ_1 and ζ_3 are 1, while their phases θ_1 and θ_3 are related by $\theta_3 = \pi - \frac{\theta_1}{2}$. So, while θ_1 runs between 0 and π, θ_3 spans the interval between π and $\pi/2$. These two angular intervals are mapped to the same curve in the target complex plane: the curved part of the lobe boundary in $D_1^{(+)}$ and a piece of the curved boundary of $D_3^{(+)}$. Finally $f_2^{(+)}(\zeta_2) = f_3^{(+)}(\zeta_3)$ if $\zeta_2 = -1/\zeta_3^2$. That means the moduli of ζ_2 and ζ_3 are 1, while their phases θ_2 and θ_3 are related by $\theta_2 = \pi - 2\theta_3$. So, while θ_2 runs between 0 and π, θ_3 spans the interval bewtween
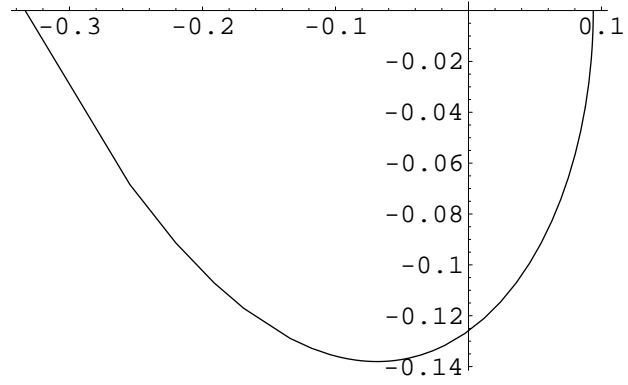
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Figure 2: *The region $D_2^{(+)}$ is contained between the curve and the real axis.*
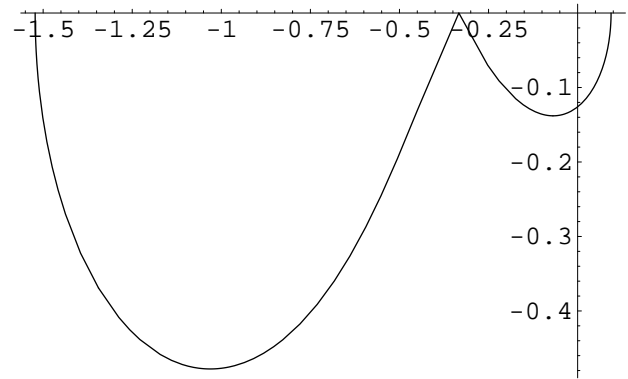

Figure 3: *The region $D_3^{(+)}$ is the portion of the lower half plane external to the curve.*

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$\pi/2$ and 0. Again these two angular intervals are mapped to the same curve in the target complex plane: the curved boundary of $D_2^{(+)}$ and the other piece of the curved boundary of $D_3^{(+)}$. In a similar way one can proceed with the remaining three correspondences.

So far the discussion has been purely mathematical, without any concern for the string process we want to describe. From the point of view of string theory the 'physical' branches are the ones where the expansion for small x_r is of the form $\varphi_r = 1 + x_r + \dots$, i.e. the $+$ branches in (31–33). So we have to glue together the images of the semidisks S_1, S_2, S_3 by $f_1^{(+)}, f_2^{(+)}, f_3^{(+)}$ respectively, according to the maps (23), identifying the common boundaries. This means that we have to glue the inner part of the lobe $D_1^{(+)}$ and $D_2^{(+)}$ with $D_3^{(+)}$, i.e. with the outer part of the bi–lobe. The result is the entire lower half plane. The process described is the joining of the strings 1 and 2 by two endpoints to form the third string at the interaction point $\zeta_1 = -1, \zeta_2 = 1$ and $\zeta_3 = i$. The picture obtained in this way allows us to make a comparison with the three string interaction in covariant SFT. As one can see there are remarkable differences. In particular the midpoints of the three strings never overlap (at variance with covariant OSFT); on the contrary, the LCSFT vertex preserves its characteristic perturbative geometry where the strings interact by the endpoints.

For completeness we write down the explicit expressions of the generating functions for the Neumann coefficients. In these definitions we always use the $+$ branch of φ_r

$$
N^1(z) = \ln\frac{1 + \sqrt{1 + \frac{1}{z}}}{2}, \quad N^2(z) = -N^1(-z), \quad N^3(z) = 2\ln\left(\frac{1}{z} + \sqrt{1 + \frac{1}{z^2}}\right) \quad (37)
$$

and

$$
N^{11}(z_1, z_2) = \ln\left(\frac{1}{2(z_1 - z_2)}\left(\frac{1}{1 - \sqrt{1 + \frac{1}{z_2}}} - \frac{1}{1 - \sqrt{1 + \frac{1}{z_1}}}\right)\right),
$$

$$
N^{22}(z_1, z_2) = N^{11}(-z_1, -z_2),
$$

$$
N^{33}(z_1, z_2) = \ln\left(\frac{1}{z_1 - z_2}\left(\sqrt{1 + z_1^2} - \sqrt{1 + z_2^2}\right)\right) \quad (38)
$$

as well as

$$
N^{12}(z_1, z_2) = -N^{11}(z_1, -z_2),
$$

$$
N^{23}(z_1, z_2) = \ln\left(\frac{\sqrt{1 - \frac{1}{z_1}} - \sqrt{1 + z_2^2}}{1 - \sqrt{1 + z_2^2}}\right),
$$

$$
N^{13}(z_1, z_2) = \ln\left(\frac{\sqrt{1 + \frac{1}{z_1}} + \sqrt{1 + z_2^2}}{1 + \sqrt{1 + z_2^2}}\right), \quad N^{rs}(z_1, z_2) = N^{sr}(z_2, z_1). \quad (39)
$$

Finally let us remark that, for $\beta \neq 1$, the conformal mappings f_r may be far more complicated than in the above example. With β rational they have a finite number of

branches that describe a Riemann surface. Like in the $\beta = 1$ case, only one of them will define the correct Neumann coefficients. When β is irrational, as shown in (18), this branch can always be determined, but the Riemann surface interpretation is lost.

3   The dTL Hirota equations

Let us introduce the flow parameters $t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}$ with $k = 1, 2, .., \infty$ and the differential operators

$$D_r(z) = \sum_{k=1}^{\infty} \frac{1}{kz^k} \frac{\partial}{\partial t_k^{(r)}}, \qquad \bar{D}_r(\bar{z}) = \sum_{k=1}^{\infty} \frac{1}{k\bar{z}^k} \frac{\partial}{\partial \bar{t}_k^{(r)}}, \qquad r, s = 1, 2, 3 \quad modulo \quad 3 . \quad (40)$$

The Hirota equations for the three decoupled copies of the dispersionless Toda lattice hierarchies fit for the present case are (for more details, see [6] and references therein)

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F} = z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F} - z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}, \qquad (41)$$

$$(\bar{z}_1 - \bar{z}_2)e^{\bar{D}_r(\bar{z}_1)\bar{D}_r(\bar{z}_2)F} = \bar{z}_1 e^{-\partial_{t_0^{(r)}}\bar{D}_r(\bar{z}_1)F} - \bar{z}_2 e^{-\partial_{t_0^{(r)}}\bar{D}_r(\bar{z}_2)F}, \qquad (42)$$

$$-z_1\bar{z}_2\left(1 - e^{-D_r(z_1)\bar{D}_r(\bar{z}_2)F}\right) = e^{\partial_{t_0^{(r)}}(\partial_{t_0^{(r)}} + D_r(z_1) + \bar{D}_r(\bar{z}_2))F} \qquad (43)$$

where $F \equiv F(\{t_0^{(r)}, t_k^{(r)}, \bar{t}_k^{(r)}\})$ is the τ–function (free energy) of the hierarchy. The minus sign on the l.h.s. of eq. (43) can be replaced by the standard plus sign via the transformations $\{t_n^r, \bar{t}_n^{\ r}\} \Rightarrow \{t_n^r, (-1)^n \bar{t}_n^{\ r}\}$ or $\{t_n^r, \bar{t}_n^{\ r}\} \Rightarrow \{(i)^n t_n^r, (i)^n \bar{t}_n^{\ r}\}$.

We remark that eqs. (41–43) form three distinct sets of equations, each one being formally the same as in Witten's OSFT, [1]. The fact is that in the latter case these sets of equations collapse to the same set.

Now, it is elementary to verify that the generating functions $N^r(z)$, $N^{rs}(z_1, z_2)$ (15–17) of the Neumann coefficients satisfy the following equations:

$$\begin{aligned}
(z_1 - z_2)e^{N^{rr}(z_1,z_2)} &= z_1 e^{-\beta_{r+1}\beta_r N^r(z_1)} - z_2 e^{-\beta_{r+1}\beta_r N^r(z_2)} \\
&= z_1 e^{\beta_r N^r(z_1)} - z_2 e^{\beta_r N^r(z_2)}
\end{aligned} \qquad (44)$$

and

$$-z_1\bar{z}_2\left(1 - e^{N^{rr+1}(z_1,\bar{z}_2)}\right) = e^{\tau_0(\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}) + \beta_{r+1}\beta_r N^r(z_1) - \beta_{r+1} N^{r+1}(\bar{z}_2)}. \qquad (45)$$

These are easily seen to reproduce the Hirota equations (41–43) provided we suitably identify the generating functions with the second derivatives of F as follows:

$$\begin{aligned}
D_r(z_1)D_r(z_2)F &= N^{rr}(z_1, z_2), \\
\bar{D}_r(\bar{z}_1)\bar{D}_r(\bar{z}_2)F &= N^{r+1r+1}(\bar{z}_1, \bar{z}_2), \\
D_r(z_1)\bar{D}_r(\bar{z}_2)F &= -N^{rr+1}(z_1, \bar{z}_2),
\end{aligned}$$

$$\partial_{t_0^{(r)}} D_r(z)F = \beta_{r+1}\beta_r N^r(z),$$

$$\partial_{t_0^{(r)}} \bar{D}_r(\bar{z})F = -\beta_{r+1}N^{r+1}(\bar{z}),$$

$$\partial_{t_0^{(r)}}\partial_{t_0^{(r)}}F = \tau_0\Big(\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}\Big). \tag{46}$$

However we remark that we can also define a consistent reduction by:

$$\frac{\partial}{\partial \bar{t}_n^{(r)}}F = \pm\frac{\partial}{\partial t_n^{(r+1)}}F. \tag{47}$$

This allows us to define new identifications as follows

$$D_r(z_1)D_r(z_2)F = N^{rr}(z_1, z_2),$$

$$D_r(z_1)D_{r+1}(z_2)F = \mp N^{rr+1}(z_1, z_2),$$

$$\partial_{t_0^{(r)}} D_r(z)F = \beta_{r+1}\beta_r N^r(z),$$

$$\partial_{t_0^{(r)}} D_{r+1}(z)F = \mp\beta_{r+1}N^{r+1}(z),$$

$$\partial_{t_0^{(r)}}\partial_{t_0^{(r)}}F = \tau_0\Big(\frac{1}{\alpha_r} + \frac{1}{\alpha_{r+1}}\Big). \tag{48}$$

The corresponding equations for the tau function are

$$
\begin{aligned}
(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F} &= z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F} - z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F} \\
&= z_1 e^{\mp\partial_{t_0^{(r-1)}}D_r(z_1)F} - z_2 e^{\mp\partial_{t_0^{(r-1)}}D_r(z_2)F}, \\
-z_1 z_2\left(1 - e^{\mp D_r(z_1)D_{r+1}(z_2)F}\right) &= e^{\partial_{t_0^{(r)}}(\partial_{t_0^{(r)}} + D_r(z_1)\pm D_{r+1}(z_2))F}.
\end{aligned} \tag{49}
$$

These Hirota equations refer to the dispersionless hierarchy produced by the cyclic (coupled) reduction (47) of the three copies of the dispersionless Toda lattice hierarchies. Our conjecture is that the resulting hierarchy can be related to the 3-punctures Whitham hierarchy [14] and the dispersionless limit of the 3-component KP hierarchy, but the detailed analysis of this point is out of the scope of the present paper and will be given elsewhere.

4 Hirota equations for more general correlators

As we will see in the next section, in order to verify the validity of the full Hirota equations we need to know correlators with more than two entries. The dispersionless Hirota equations for 'n–point functions' are obtained by differentiating $n-2$ times the equations (41,42) and (43). These derived Hirota equations, like the original ones, do not uniquely determine their solutions. In order to be able to write down the latter we have to provide some additional information. In fact what we are looking for are n–point correlators that are compatible with the Neumann coefficients of the previous section and satisfy the full Hirota equation (see next section). Although we don't have a proof of it, we believe these two requirements completely determine the full series of n–point functions.

In this section we would like to concentrate on three– and four–point functions. On the basis of what we have just said, in order to determine them we have to rely on plausibility arguments and verify the results *a posteriori*. We notice first that the Neumann coefficients for the three strings vertex can be interpreted in terms of two–point correlators of a 1D system. For instance

$$\langle V_3|a_m^{(r)I\dagger}a_n^{(s)J\dagger}|0\rangle = \delta^{IJ}\,V_{mn}^{rs} = -\sqrt{nm}\,N_{mn}^{rs}\,\delta^{IJ}\ . \tag{50}$$

Analogous relations hold for two–point functions involving zero modes. In the same way we can of course consider correlators with more insertions. It is obvious that all the correlators with an odd number of insertions identically vanish. We take these correlators as a model in order to calculate three– and four–point functions of a quantum system that satisfies the full Hirota equations. We will see below that the correct answer is not the three– and four–point correlators obtained by inserting one and two creation operators in (50), respectively, but combinations of them. We will refer to the underlying model, based on the two–point correlators (50) and satisfying the Hirota equation, as the *associated quantum system*. It must be clear that we do not know yet what this system is, we simply postulate its existence. In particular, as a working hypothesis, we assume that its three–point functions identically vanish.

The dispersionless Hirota equations for four–point functions are obtained by differentiating twice the equations (41,42) and (43). Of course we have the possibility of applying $\partial_{t_0^{(r)}}$ or $D_r(z)$. Therefore from (41), for instance, we obtain

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\prod_{i=1}^{4}D_r(z_i)F = \tag{51}$$

$$z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial_{t_0^{(r)}}D_r(z_2)\prod_{i=3}^{4}D_r(z_i)F - z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial_{t_0^{(r)}}D_r(z_1)\prod_{i=3}^{4}D_r(z_i)F,$$

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\partial_{t_0^{(r)}}\prod_{i=1}^{3}D_r(z_i)F = \tag{52}$$

$$z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial_{t_0^{(r)}}^2 D_r(z_2)D_r(z_3)F - z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial_{t_0^{(r)}}^2 D_r(z_1)D_r(z_3)F,$$

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F}\partial_{t_0^{(r)}}^2 D_r(z_2)D_r(z_1)F = \tag{53}$$

$$z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F}\partial_{t_0^{(r)}}^3 D_r(z_2)F - z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F}\partial_{t_0^{(r)}}^3 D_r(z_1)F$$

where we have assumed that three–point correlators vanish.

After expanding the above equations in powers of z_1 and z_2, and collecting terms of the form $z_1^m z_2^n$ we obtain equations involving two and four derivative of F. We already know how to identify the two–derivative terms in terms of the Neumann coefficients of LCSFT (see previous section). The task now is to try to make the corresponding identifications for the four–derivative ones. The solutions to eqs. (51–53) are not uniquely defined (see

below). The solutions we are interested in are as follows:

$$F_{t_0^{(r)} t_0^{(r)} t_0^{(r)} t_m^{(r)}} = -3(\alpha_r)^2 \frac{1+\beta_r}{\beta_r} m(N_{m0}^{rr+1} - N_{m0}^{rr}),$$

$$F_{t_0^{(r)} t_0^{(r)} t_m^{(r)} t_n^{(r)}} = -(\alpha_r)^2 mn \left[\frac{1+\beta_r}{\beta_r}(n+m)N_{mn}^{rr} + 2(N_{m0}^{rr+1} - N_{m0}^{rr})(N_{n0}^{rr+1} - N_{n0}^{rr}) \right],$$

$$F_{t_0^{(r)} t_m^{(r)} t_n^{(r)} t_l^{(r)}} = -(\alpha_r)^2 mnl[(n+l)(N_{m0}^{rr+1} - N_{m0}^{rr})N_{nl}^{rr} + (n+m)(N_{l0}^{rr+1} - N_{l0}^{rr})N_{mn}^{rr}$$

$$+ (m+l)(N_{n0}^{rr+1} - N_{n0}^{rr})N_{ml}^{rr}], \tag{54}$$

$$F_{t_m^{(r)} t_n^{(r)} t_l^{(r)} t_k^{(r)}} = -(\alpha_r)^2 mnlk[(m+n)(l+k)N_{mn}^{rr}N_{lk}^{rr} + (m+l)(n+k)N_{ml}^{rr}N_{nk}^{rr}$$

$$+ (m+k)(n+l)N_{mk}^{rr}N_{nl}^{rr}]$$

where N_{m0}^{rs} are defined in eqs. (3). Multiplying these by the appropriate monomials of z variables and summing, leads to the following compact expressions (which will be used later on):

$$\partial_{t_0^{(r)}}^3 D_r(z)F = \frac{3\alpha_{r-1}^2}{\beta_r} \frac{\varphi_r(z^{-1}) - 1}{(1+\beta_r)\varphi_r(z^{-1}) - \beta_r}, \tag{55}$$

$$\partial_{t_0^{(r)}}^2 \prod_{i=1}^{2} D_r(z_i)F = -3\alpha_{r-1}^2 \prod_{i=1}^{2} \frac{\varphi_r(z_i^{-1}) - 1}{(1+\beta_r)\varphi_r(z_i^{-1}) - \beta_r},$$

$$\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = 3\alpha_{r-1}^2 \beta_r \prod_{i=1}^{3} \frac{\varphi_r(z_i^{-1}) - 1}{(1+\beta_r)\varphi_r(z_i^{-1}) - \beta_r},$$

$$\prod_{i=1}^{4} D_r(z_i)F = -3\alpha_{r-1}^2 \beta_r^2 \prod_{i=1}^{4} \frac{\varphi_r(z_i^{-1}) - 1}{(1+\beta_r)\varphi_r(z_i^{-1}) - \beta_r}.$$

Using these expressions one can easily verify that they satisfy equations (51), (52) and (53).

A comment is in order concerning the results we have just written down. We have already said that eqs. (55) are not unique solutions to the Hirota equations for four–point functions. We have single them out because they are compatible with the dispersive Hirota equations (see next section) and they will allow us to find a solution of the latter coherent with the dispersionless Neumann coefficients. We recall that they are also compatible with vanishing three–point functions. As was mentioned at the beginning of this section, at first sight it would seem that the two–point functions, three–point functions and four–point functions (i.e. the derivatives of F with respect to two, three and four t_n parameters) of our system are simply given by

$$\langle V_3 | a_{n_1}^{r_1\dagger} \dots a_{n_k}^{r_k\dagger} | 0 \rangle. \tag{56}$$

While this is true for two– and three–point functions, it is not quite true for the four–point ones, as one can see by comparing (56) with (54). It is evident that the four–point functions given by (56) has the right form, but must be suitably combined in order to

coincide with (54), which satisfy the Hirota equations. Therefore the correlators of the associated quantum system that underlies the Neumann coefficients of LCSFT are made of suitable combinations of the k–point functions (56). In conclusion: we have found a solution to the Hirota equations for the four–point functions, which is compatible with the Mandelstam's three strings vertex of the LCSFT (and with its deformations up to second order in the expansion parameter, see next section), however we do not have a proof that this solution is unique.

A discussion of the general solution to eqs. (51–53) can be found in Appendix A.

5   PP–wave SFT and the dispersive Hirota equations

String theory on a maximally supersymmetric pp–wave background (plane wave limit of $AdS_5 \times S^5$) [15, 16], is exactly solvable [17]. Building on this it has been recently possible to construct the exact three strings vertex for the LCSFT on this background, i.e. to completely specify the relevant Neumann coefficients. They depend on the 'string mass' μ (determined by the five form flux of type IIB superstring theory). When $\mu \to 0$ one recovers Mandelstam's Neumann coefficients discussed in the previous sections. In other words the nontrivial string background deforms the Neumann coefficients. It is interesting to see whether the Hirota equations discussed in the previous sections get deformed accordingly in such a way as to preserve integrability. This is our guess and this is what we would like to provide evidence for in the remaining part of our paper.

Let us start from the expression of the μ–deformed three strings vertex, [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. The bosonic part, to which we limit ourselves[1], is defined by

$$\Delta'_B = \frac{1}{2}\delta_{IJ}\left(\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty} a_m^{(r)I\dagger}\mathcal{V}_{mn}^{rs}a_n^{(s)J\dagger}\right) \ . \tag{57}$$

The commutation relations are as in section 2, but $a_m^{(r)I\dagger} \neq a_{-m}^{(r)I}$ and $a_m^{(r)I}|0\rangle = 0$ for $m \in \mathbb{Z}$. The coefficients with negative n, m labels can be obtained from those with positive ones[2]. In this paper we consider only the $\mathcal{V}_{mn}^{rs}$ with $m, n \geq 0$. We set

$$\mathcal{V}_{mn}^{rs} = -\sqrt{mn}\,\mathcal{N}_{mn}^{rs} \ . \tag{58}$$

The Neumann coefficients $\mathcal{N}_{mn}^{rs}$ with $m, n \geq 1$ have been calculated in an explicit way in

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[1] For the problems connected with the supersymmetric completion of the vertex and its prefactor see the reviews [28, 29, 30] and references therein.

[2] There is a subtle point here. The expressions of Δ'_B and Δ_B in eq.(2) for the three strings vertex refer to two different bases: Δ_B is expressed in terms of the transverse momenta p_r^I (momentum basis) while Δ'_B contains the zero mode oscillators $a_0^{(r)I}$ (oscillator basis). This means that starting from (1) we have passed from the former basis to the latter by explicitly integrating over the transverse momenta (see, for instance, [31]). This operation modifies in a well–known way the vertex coefficients. However, in the present case, it turns out that in the limit $\mu \to 0$ the coefficients $\mathcal{V}_{mn}^{rs}$ tend to the corresponding coefficients V_{mn}^{rs} in (2) at least for $m, n \geq 1$.

[26, 27]:

$$\mathcal{N}_{mn}^{rs} = -\frac{mn\alpha}{\alpha_s \omega_{r,m} + \alpha_r \omega_{s,n}} \frac{\mathcal{N}_m^r \mathcal{N}_n^s}{\alpha_r \alpha_s} \tag{59}$$

where $\omega_{r,m} = \sqrt{m^2 + \alpha_r^2 \mu^2}$ and

$$\mathcal{N}_m^r = \sqrt{\frac{\omega_{r,m}}{m}} \frac{\omega_{r,m} + \alpha_r \mu}{m} f_m^{(r)} \;. \tag{60}$$

In these formulas

$$f_m^{(r)} = -\frac{e^{\tau_0 \omega \frac{m}{\alpha_r}}}{m(\alpha_r + \alpha_{r+1})\omega_{\frac{m}{\alpha_r}}} \frac{\Gamma_\mu^{(r+1)}\left(-\frac{m}{\alpha_r}\right)}{\Gamma_\mu^{(r)}\left(\frac{m}{\alpha_r}\right)\Gamma_\mu^{(r-1)}\left(\frac{m}{\alpha_r}\right)} \tag{61}$$

and $\omega_z = \text{sgn}(z)\sqrt{\mu^2 + z^2}$. The μ–deformed Γ functions are defined by

$$\Gamma_\mu^{(r)}(z) = \frac{e^{-\gamma \alpha_r \omega_z}}{\alpha_r z} \prod_{n=1}^{\infty} \left(\frac{n}{\omega_{r,n} + \alpha_r \omega_z} e^{\frac{\alpha_r \omega_z}{n}}\right) \tag{62}$$

where γ is the Euler–Mascheroni constant. It must be remarked that the above coefficients are not written in the usual form, we have simplified them by dropping some intermediate inessential factors.

Here we present some results which are going to be needed in the next sections. Expanding (60) and (59) in powers of μ up to order μ^2 we obtain

$$\mathcal{N}_m^r = N_m^r \left[1 + \frac{\alpha_r}{m}\mu + \left(\frac{\alpha_r^2}{4m^2} + \frac{\alpha_r}{2m}\tau_0\right)\mu^2 + ...\right] \tag{63}$$

and

$$\mathcal{N}_{mn}^{rs} = N_{mn}^{rs} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\mu + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n} + 2\tau_0\right)\mu^2 + ...\right] \tag{64}$$

where N_m^r and N_{mn}^{rs} are Mandelstam's Neumann coefficients. For reasons that will become clear in the next section, it is convenient to rewrite these expansions in powers of λ related to μ as: $\mu = \lambda - \frac{\tau_0}{2}\lambda^2 + ...$

$$\mathcal{N}_m^r = N_m^r \left[1 + \frac{\alpha_r}{m}\lambda + \left(\frac{\alpha_r^2}{4m^2}\right)\lambda^2 + ...\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}_{j,m}^r \lambda^j, \tag{65}$$

$$\mathcal{N}_{mn}^{rs} = N_{mn}^{rs} \left[1 + \left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)\lambda + \frac{1}{4}\left(\frac{\alpha_r}{m} + \frac{\alpha_s}{n}\right)^2 \lambda^2 + ...\right] \equiv \sum_{j=0}^{\infty} \mathcal{N}_{j,mn}^{rs} \lambda^j \tag{66}$$

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where the subscript j refers to the order of expansion in λ.

Again, for later use, it is convenient to organize these Neumann coefficients by means of associated generating functions

$$\mathcal{N}_j^r(z) := \sum_{m=1}^{\infty} \frac{1}{z^m} \, m^{2j} \, \mathcal{N}_{j,m}^r, \tag{67}$$

$$\mathcal{N}_j^{rs}(z_1, z_2) := \sum_{m,n=1}^{\infty} \frac{1}{z_1^m} \frac{1}{z_2^n} \, (mn)^j \, \mathcal{N}_{j,mn}^{rs}, \tag{68}$$

for $j = 0, 1, 2$. At each order the summation can easily be carried out to give

$$\mathcal{N}_1^r(z) = \alpha_r \frac{\varphi_r(z^{-1}) - 1}{(1 + \beta_r)\varphi_r(z^{-1}) - \beta_r}, \tag{69}$$

$$\mathcal{N}_2^r(z) = \frac{\alpha_r^2}{4} \frac{\varphi_r(z^{-1})(\varphi_r(z^{-1}) - 1)}{((1 + \beta_r)\varphi_r(z^{-1}) - \beta_r)^3}, \tag{70}$$

$$\mathcal{N}_1^{rs}(z_1, z_2) = -\frac{\alpha}{\alpha_r \alpha_s} \frac{\varphi_r(z_1^{-1}) - 1}{(1 + \beta_r)\varphi_r(z_1^{-1}) - \beta_r} \frac{\varphi_s(z_2^{-1}) - 1}{(1 + \beta_s)\varphi_s(z_2^{-1}) - \beta_s}, \tag{71}$$

$$\mathcal{N}_2^{rs}(z_1, z_2) = -\frac{\alpha}{4\alpha_r \alpha_s} \frac{(\varphi_r(z_1^{-1}) - 1)(\varphi_s(z_2^{-1}) - 1)}{[(1 + \beta_r)\varphi_r(z_1^{-1}) - \beta_r][(1 + \beta_s)\varphi_s(z_2^{-1}) - \beta_s]}$$
$$\times \left[\frac{\alpha_s \varphi_r(z_1^{-1})}{[(1 + \beta_r)\varphi_r(z_1^{-1}) - \beta_r]^2} + \frac{\alpha_r \varphi_s(z_2^{-1})}{[(1 + \beta_s)\varphi_s(z_2^{-1}) - \beta_s]^2} \right]. \tag{72}$$

We notice that in order to get these compact generating functions it is necessary to insert the m^{2j} and $(mn)^j$ factors in (67) and (68), respectively.

5.1 The dispersive Hirota equations

Here, we consider the full (dispersive) Hirota equations for the Toda lattice hierarchy (see, e.g. ref. [6] and references therein). For the sake of brevity we deal in the sequel with half of the story, namely only with those equations that involve unbarred z variables (corresponding to eq.(41))

$$z_1 \left(e^{\lambda(\partial_{t_0} - D_r(z_1))} \tau_\lambda \right) \left(e^{-\lambda D_r(z_2)} \tau_\lambda \right) - z_2 \left(e^{\lambda(\partial_{t_0} - D_r(z_2))} \tau_\lambda \right) \left(e^{-\lambda D_r(z_1)} \tau_\lambda \right) \tag{73}$$
$$= (z_1 - z_2) \left(e^{-\lambda(D_r(z_1) + D_r(z_2))} \tau_\lambda \right) \left(e^{\lambda \partial_{t_0}} \tau_\lambda \right)$$

where λ is a deformation parameter and τ_λ is the full *tau function* of the Toda lattice (KP) hierarchy:

$$\tau_\lambda = \exp(\mathcal{F}_\lambda), \qquad \mathcal{F}_\lambda = \frac{1}{\lambda^2} F_0 + \frac{1}{\lambda} F_1 + F_2 + \ldots \quad . \tag{74}$$

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In order to find a solution to (73) it is useful to proceed in two steps, and solve first the Hirota equation that does not involve t_0 derivatives (corresponding to the KP hierarchy), that is

$$(z_1 - z_2)\left(e^{-\lambda(D_r(z_1)+D_r(z_2))}\tau_\lambda\right)\left(e^{-\lambda D(z_3)}\tau_\lambda\right) + \text{cycl. perms. of} \quad 1,2,3 = 0. \tag{75}$$

Our conjecture is that these equations are obeyed by the Neumann coefficients $\mathcal{N}_{mn}^{rs}$ introduced above, provided we identify λ with a suitable function $f(\mu)$ of μ. At present we are not in the condition to prove this conjecture in a non–perturbative way. But we can expand in powers of λ and try to prove it order by order in λ. We will be able to do it up to second order in λ, with the identification $\lambda = \mu + \frac{\tau_0}{2}\mu^2 + \dots$.

Expanding (73) in powers of λ we obtain an infinite set of equations that constrain the correlators at the different orders of approximation. To order 0 we get the dispersionless Hirota equation. Our task is therefore to prove that the next order equations hold.

5.2   Order μ approximation

Expanding (75) to first order in λ and identifying λ with μ we find

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}D_r(z_1)D_r(z_2)F_1 + \text{cycl. perms. of} \quad 1,2,3 = 0 \tag{76}$$

while, expanding (73), we get

$$- z_1 e^{-\partial_{t_0^{(r)}}D_r(z_1)F_0}\partial_{t_0^{(r)}}D_r(z_1)F_1 + z_2 e^{-\partial_{t_0^{(r)}}D_r(z_2)F_0}\partial_{t_0^{(r)}}D_r(z_2)F_1$$

$$= (z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}D_r(z_1)D_r(z_2)F_1. \tag{77}$$

In deriving these equations we have used the information that all third order derivatives of F_0 (three–point functions) vanish (see previous section). It is easy to see that if we make the following identifications:

$$D_r(z_1)D_r(z_2)F_1 = \mathcal{N}_1^{rr}(z_1, z_2) \tag{78}$$

with $D_r(z_1)D_r(z_2)F_0$ as in (46), eq.(76) is satisfied. If in addition we identify

$$\partial_{t_0^{(r)}}D_r(z)F_1 = \beta_{r+1}\beta_r\mathcal{N}_1^r(z), \tag{79}$$

with $\partial_{t_0^{(r)}}D_r(z)F_0$ as in (46), the more general eq.(77) is satisfied as well.

5.3   Order μ^2 approximation

Expanding (75) to order λ^2 and identifying λ with $\mu + \frac{\tau_0}{2}\mu^2 + \dots$ we find

$$(z_1 - z_2)e^{D_r(z_1)D_r(z_2)F_0}\left(D_r(z_1)D_r(z_2)F_2 + \frac{1}{2}\left(D_r(z_1)D_r(z_2)F_1\right)^2 + \frac{1}{6}D_r(z_1)D_r(z_2)^3F_0\right.$$

$$\left. + \frac{1}{4}D_r(z_1)^2D_r(z_2)^2F_0 + \frac{1}{6}D_r(z_1)^3D_r(z_2)F_0\right) + \text{cycl. perms. of } 1,2,3 = 0 \tag{80}$$

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while expanding (73) to the same order we get

$$z_1 e^{-\partial_{t_0^{(r)}} D_r(z_1) F_0} \left(-\partial_{t_0^{(r)}} D_r(z_1) F_2 + \frac{1}{2} (\partial_{t_0^{(r)}} D_r(z_1) F_1)^2 + \frac{1}{4} \partial^2_{t_0^{(r)}} D_r(z_1)^2 F_0 - \frac{1}{6} \partial_{t_0^{(r)}} D_r(z_1)^3 F_0 \right.$$

$$\left. - \frac{1}{6} \partial^3_{t_0^{(r)}} D_r(z_1) F_0 \right) - z_2 e^{-\partial_{t_0^{(r)}} D_r(z_2) F_0} \left(-\partial_{t_0^{(r)}} D_r(z_2) F_2 + \frac{1}{2} (\partial_{t_0^{(r)}} D_r(z_2) F_1)^2 \right.$$

$$\left. + \frac{1}{4} \partial^2_{t_0^{(r)}} D_r(z_2)^2 F_0 - \frac{1}{6} \partial_{t_0^{(r)}} D_r(z_2)^3 F_0 - \frac{1}{6} \partial^3_{t_0^{(r)}} D_r(z_2) F_0 \right)$$

$$= (z_1 - z_2) e^{D_r(z_1) D_r(z_2) F_0} \left(D_r(z_1) D_r(z_2) F_2 + \frac{1}{2} (D_r(z_1) D_r(z_2) F_1)^2 + \frac{1}{6} D_r(z_1) D_r(z_2)^3 F_0 \right.$$

$$\left. + \frac{1}{4} D_r(z_1)^2 D_r(z_2)^2 F_0 + \frac{1}{6} D_r(z_1)^3 D_r(z_2) F_0 \right). \tag{81}$$

In deriving this equation we have used once again the information that all odd order derivatives of F_0 and F_1 vanish. Eq.(80) is satisfied provided we make the following identifications:

$$D_r(z_1) D_r(z_2) F_2 = \mathcal{N}_2^{rr}(z_1, z_2) \tag{82}$$

with four-derivatives as defined in (55). If, in addition, we identify

$$\partial_{t_0^{(r)}} D_r(z) F_2 = \beta_{r+1} \beta_r \mathcal{N}_2^r(z) \tag{83}$$

the more general equation (81) is satisfied.

Taking into account eqs. (46), (78–79) and (82–83) it is plausible to expect that the following universal identification is valid in general:

$$D_r(z_1) D_r(z_2) F_j = \mathcal{N}_j^{rr}(z_1, z_2), \quad \partial_{t_0^{(r)}} D_r(z) F_j = \beta_{r+1} \beta_r \mathcal{N}_j^r(z), \quad j = 0, 1, 2, 3, ... \tag{84}$$

although in this case one may have to modify the definitions (67-68) of the generating functions for $j > 2$.

6 Discussion

In this paper, to start with, we have shown that the three strings vertex coefficients (or, more properly, the corresponding Neumann coefficients) in LCSFT obey the Hirota equations for the dispersionless Toda lattice hierarchy. We have then written down the Hirota equations for the four–point functions and found solutions consistent with identically vanishing three–point functions. Finally we have expanded the three strings vertex of the LCSFT in a maximally supersymmetric pp–wave background in a series in the μ parameter, and, in a parallel way, we have expanded the Hirota equations for the full Toda lattice hierarchy in the deformation parameter. After a (non–trivial) identification of the two parameters we have been able to show that the latter are satisfied by the former up to

second order. The calculations beyond second order become technically very challenging and it is clear that a non–perturbative approach is needed for a satisfactory proof that LCSFT three–strings vertex obey the Hirota equations for the full Toda lattice hierarchy.

We would like to end this paper with a few open questions. The first concerns the type of hierarchy underlying the three strings vertex of LCSFT. We have always mentioned the Toda lattice hierarchy, but one should be more precise. Looking at section 3 it is evident that one has to do not with one Toda lattice hierarchy but rather with three coupled copies of the Toda lattice hierarchy. As we mentioned in the same section, there is the possibility of a consistent reduction. A precise formulation and identification of the relevant reduced hierarchies is a task we have not tackled in this paper (see, however, the remark at the end of section 3).

As we have pointed out in section 4, the four–point functions we have found are not the unique solutions to the Hirota equations for four–point functions. The one that we have found however are likely to be the unique solution that are consistent with the two–point functions of order 0, 1 and 2 in the μ expansion (although we have not been able to exclude other solutions). This question is intertwined with two related problems: on the one hand the question of defining in terms of matter oscillators $a_n^{(r)}$ the 1D associated quantum system whose two–point correlators are given by $\langle V_3|a_m^{(r)\dagger}a_n^{(s)\dagger}|0\rangle$ and underlies the LCSFT three strings vertex[3]; on the other hand defining an integrable system, very likely a matrix model, where all these correlators can explicitly be calculated. A successful search of such an integrable model is also likely to lead us to a natural explanation of why the somewhat mysterious factors m^{2j} and $(nm)^j$ need to be inserted in the generating functions (67–68) in order to square matters.

Appendix A. A discussion of four–point functions

Using Hirota equations (51–53) one can easily express the four–point functions $\prod_{i=1}^4 D_r(z_i)F$, $\partial_{t_0^{(r)}}\prod_{i=1}^3 D_r(z_i)F$ and $\partial_{t_0^{(r)}}^2\prod_{i=1}^2 D_r(z_i)F$ in terms of the two–point functions $\prod_{i=1}^2 D_r(z_i)F$ and $\partial_{t_0^{(r)}}D_r(z)F$ as well as the four–point function $\partial_{t_0^{(r)}}^3 D_r(z)F$. The two–point functions were already identified with the corresponding generating functions

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[3]We have seen in section 4 that the four–point functions that are required by the Hirota equations are not obtained by simple adjunction of two creation operators in this expression (see the final remark there).

of the Neumann coefficients (see, eqs. (46)), so one is left with the problem of first finding a suitable identification for the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$, which leads to the straightforward derivation of all other. It is interesting that if one additionally requires that the four–point functions admit a factorizable form as functions of z_i, then the identification of $\partial^3_{t_0^{(r)}} D_r(z)F$ is uniquely fixed modulo three arbitrary $SL(2)$-group parameters a_r, b_r, c_r and d_r ($b_r c_r - a_r d_r = 1$) as follows

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})} \frac{a_r \varphi_r(z^{-1}) + b_r}{c_r \varphi_r(z^{-1}) + d_r} \ . \tag{85}$$

Using this and eqs. (51–53), one can derive the corresponding explicit expressions for the remaining four–point functions

$$\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F = -\prod_{i=1}^{2} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \ , \tag{86}$$

$$\partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = -d_r \prod_{i=1}^{3} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \ , \tag{87}$$

$$\prod_{i=1}^{4} D_r(z_i)F = -d_r^2 \prod_{i=1}^{4} \frac{\varphi_r(z_i^{-1}) - 1}{c_r \varphi_r(z_i^{-1}) + d_r} \ . \tag{88}$$

The four–point functions (55) are a particular case of the latter for

$$d_r = -\frac{1}{a_r} = -\frac{\beta_r}{\sqrt{3}\,\alpha_{r-1}}, \quad b_r = 0, \quad c_r = -\frac{1}{\sqrt{3}\,\alpha_r}. \tag{89}$$

Let us remark that if one rescales the parameters in eqs. (85–88) as follows

$$\{a_r,\ b_r,\ c_r,\ d_r\} \quad \Rightarrow \quad \frac{1}{\epsilon}\,\{a_r,\ b_r,\ c_r,\ d_r\} \tag{90}$$

and consider the limit $\epsilon \to 0$, then the four–point function $\partial^3_{t_0^{(r)}} D_r(z)F$ preserves its form (85) with the parameters satisfying $b_r c_r - a_r d_r = 0$, so it becomes

$$\partial^3_{t_0^{(r)}} D_r(z)F = \frac{\varphi_r(z^{-1}) - 1}{\varphi_r(z^{-1})} \frac{a_r}{c_r} \ , \tag{91}$$

but all the other four–point functions degenerate and become equal to zero, $\partial^2_{t_0^{(r)}} \prod_{i=1}^{2} D_r(z_i)F = \partial_{t_0^{(r)}} \prod_{i=1}^{3} D_r(z_i)F = \prod_{i=1}^{4} D_r(z_i)F = 0$.

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